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What Can We Learn from Nonminimally Coupled Scalar Field Cosmology?

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A novel exploration of nonminimally coupled scalar field cosmology is proposed in the framework of spatially flat Friedmann–Robertson–Walker spaces for arbitrary scalar field potentials $V(\psi)$ and values of the nonminimal coupling constant ξ. This approach is self-consistent in the sense that the equation of state of the scalar field is not prescribed *a priori*, but is rather deduced together with the solution of the field equations. The role of nonminimal coupling appears to be essential. A dimensional reduction of the system of differential equations leads to the result that chaos is absent in the dynamics of a spatially flat FRW universe with a single scalar field. The topology of the phase space is studied and reveals an unexpected involved structure: according to the form of the potential $V(\psi)$ and the value of the nonminimal coupling constant ξ, dynamically forbidden regions may exist. Their boundaries play an important role in the topological organization of the phase space of the dynamical system. New exact solutions sharing a universal character are presented; one of them describes a nonsingular universe that exhibits a graceful exit from, and entry into, inflation. This behavior does not require the presence of the cosmological constant. The relevance of this solution and of the topological structure of the phase space with respect to an emergence of the universe from a primordial Minkowski vacuum, in an extended semiclassical context, is shown.

1. INTRODUCTION

The starting point of this work is the formulation and implementation of classical self-consistent cosmological histories driven by scalar fields in the framework of Einstein's theory of general relativity. Several reasons

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motivate interest in this problem; first, the necessity for a better understanding of the essential role of classical scalar fields in cosmological inflation [1] and quintessence when the fields couple nonminimally to the spacetime curvature. This is a fundamental theoretical as well as experimental problem, since present astrophysical observations [2] point with increasing confidence to an acceleration of the cosmic expansion. The latter, deduced from the data of high-redshift supernovae, is naturally modeled by scalar fields [3], and their nonminimal coupling to the Ricci curvature plays an important role in these models [4–6].

Second, scalar fields are present in most theories of high-energy physics; examples are the Higgs boson of the standard model, the string dilaton, the supersymmetric partners of spin-1/2 particles in supergravity, and the geometrical scalar field of Kaluza–Klein theories. In addition, scalar fields are expected to play a fundamental role in the physics of the early universe and drive the cosmic expansion during the inflationary era. Inflation is the only known causal mechanism which offers an explanation for the generation of density perturbations on scales above the Hubble radius [7], which evolve into the structures observed today (galaxies, clusters of galaxies, . . .). The generalization to a curved spacetime of the flat-space equation of motion for the scalar field (the Klein–Gordon equation) includes the possibility of an explicit coupling term $\xi R\psi$ between the scalar field ψ and the Ricci curvature R of spacetime. There are many reasons to believe that a nonminimal (i.e., $\xi \neq 0$) coupling term is present. In addition to the generic Callan–Coleman–Jackiw [8] argument leading to the improved energy-momentum tensor for the classical scalar field (already in flat spacetime), it appears that nonminimal coupling is generated by quantum corrections even if it is absent in the classical action [9] or is required to renormalize the theory [10]. In semiclassical gravity, where the scalar field is fully quantized while gravity is classical, it appears that conformal coupling $\xi = 1/6$ is mandatory for the realisation of a self-consistent cosmological scenario [11]. It has also been argued in quantum field theory in curved spaces that a nonminimal coupling term is to be expected whenever the spacetime curvature is large. This leads to the "ξ-problem," i.e., the problem of whether physics uniquely determines the value of ξ. The answer to this question is generally affirmative: several prescriptions for the coupling constant ξ exist and they differ according to the theory of gravity and of the scalar field adopted [12].

In general relativity (which is adopted in the present work) and in all metric theories of gravity in which the scalar field has a nongravitational origin, arguments based on the Einstein equivalence principle select the conformal value $\xi = 1/6$ [13]. The minimal ($\xi = 0$) coupling of the scalar field instead leads to physical pathologies [13]. Nonminimal couplings of the scalar field have been widely used in cosmology, and the available prescriptions have important consequences for the viability of inflationary scenarios in which the nonminimal coupling constant ξ becomes an extra parameter. The success or failure of inflation critically depends on the value of the coupling constant ξ [14, 15, 12]; the classical example is chaotic inflation with a quartic self-interaction and nonminimal coupling. For this case, even the small shift $\xi = 0 \to \xi \simeq 10^{-3}$ makes the inflationary solutions fine-tuned, and therefore the scenario becomes unviable [15].

All these reasons led numerous authors to study scalar field cosmology with nonminimal coupling in various contexts [16–20]. Its richness was nevertheless not fully explored and several of its peculiarities were missed. One reason is that the nonminimal coupling of the scalar field to the Ricci curvature adds to the energy-momentum tensor of the scalar field extra terms involving geometric quantities. A widespread attitude encountered in the literature is to disentangle this (conserved) energy-momentum tensor by including a (nonconserved) part of it in the geometric left-hand side of the Einstein equations. The resulting "new" field equations are then expressed by using the truncated remaining (nonconserved) energy-momentum tensor coupled to a redefined effective gravitational "constant" $G_{\text{eff}}(\tau) = G(1 - 8\pi G\xi\psi^2)^{-1}$, which is time dependent. G_{eff} can diverge due to a mathematically questionable division by the factor $(1 - 8\pi G\xi\psi^2)$, which can vanish; if this happens, G_{eff} is interpreted as reflecting an effective "strong" gravitational constant. The procedure introduces an artificial barrier $\psi_c = \pm(8\pi G\xi)^{-1/2}$ for $\xi > 0$ and therefore a loss of generality, since solutions which cross the barrier $\pm\psi_c$ are missed. The richness of the possible cosmological solutions associated with nonminimal coupling is therefore partially lost. On the contrary, we show how the Einstein equations, when considered in the presence of the full (conserved) energy-momentum of the nonminimally coupled scalar field, unveil these missing solutions and promote them to an unexpected and important cosmological role.

Another restrictive procedure that screens the dynamical possibilities and subtleties of the nonminimal scalar cosmological histories is the ad hoc prescription of an equation of state associated with the scalar field. Our approach, on the contrary, consists in deriving self-consistently the equation of state together with the solution of the coupled Einstein–Klein–Gordon equations, rather than to prescribe it *a priori*; hence, these solutions are called self-consistent.

A further reason motivating interest in nonminimal scalar field cosmology finds its roots in previous studies of self-consistent cosmological mechanisms in semiclassical gravity [11]. The promotion of the scalar field to a quantum nature opens unexpected avenues of approach to several cosmological problems, among which is the possible realization of nonsingular self-consistent cosmologies. The latter are possible thanks to the semiclassical mechanism of particle production induced by the expansion (or contraction) of spacetime and its feedback reaction on the rate of cosmic expansion. In the case of a scalar matter field, the Einstein semiclassical equations reduce, in the cosmological context, to a nonlinear dynamical interplay between two fields: the massive, quantum, scalar matter field, and the classical scale factor of the universe, which is the only gravitational degree of freedom compatible with the requirements of spatial homogeneity and isotropy of the universe. On one hand, this nonlinear interplay drives the cosmic expansion, and on the other hand, it regulates the production of quanta of the scalar field. It was shown [11] that the quantum production rate of massive particles induced by the cosmic expansion may lead to a feedback response which is precisely the one required to sustain the expansion. This self-consistent cooperative solution of the semiclassical Einstein equations is the de Sitter expanding solution. For this space, the dilution of the cosmic scalar fluid due to the expansion is exactly balanced by the quantum production of scalar field particles at all times. This was the first inflationary scenario proposed independently of particle physics consideration. The crucial role of the nonminimal coupling of the scalar matter field to the spacetime curvature is evident in the framework of this mechanism, as its realization necessarily requires conformal coupling ($\xi = 1/6$).

Although very appealing, this semiclassical approach is not free of ambiguities and controversies. First, there is the renormalization procedure unavoidably attached to the quantum treatment of the scalar matter field. This procedure, as well as its physical implications, is not uniquely prescribed in the time-dependent curved spacetime background [22]. In addition, the formulation of initial conditions has its conceptual as well as mathematical difficulties; in fact, these conditions are simultaneously related to classical geometrical constraints and to the choice of the corresponding quantum vacuum state of the matter field in the Heisenberg picture. The latter introduces, for example, the particle number, which is neither a “natural” Einstein equations variable nor a well-defined concept in a curved dynamical spacetime. Moreover, the semiclassical self-consistent cooperative solution is not a complete, realistic, cosmological history since it describes eternal expansion at a constant rate without spontaneous exit from inflation into a realistic expansion law (a problem common to many inflationary scenarios).

Hence the following question: is it possible to keep the self-consistent cooperative paradigmatic approach to cosmology without the quantum bewilderments? More precisely, is it possible to rephrase classically the underlying mechanism in the framework of Einstein’s classical equations with a classical scalar field as the source of gravity?

That this question may have a positive answer follows from the general structure of the mechanism driving the feedback self-consistent dynamical behavior, which is independent of the quantum or classical character of the matter field: a conservative transfer of energy, regulated by the Einstein equations, between the time-dependent spacetime geometry [the scale factor $a(t)$] and the scalar matter field $\psi(t)$. In semiclassical theory, this mechanism is achieved by the interaction of these two (classical and quantum, respectively) fields, due, for example, to the mass of the matter field. Nothing forbids *a priori* the purely classical implementation of this conservative energy transfer mechanism, which appears indeed to be contained in the Einstein equations in the cosmological context. The quantum nature of the matter scalar field does not seem to be an essential ingredient; it only gives rise to a particular implementation of the general mechanism, subject to specific constraints imposed by quantum field theory in curved spacetimes.

The structure of the Einstein and Klein–Gordon equations relating the spacetime geometry with the matter field energy-momentum are formally the same in both the classical and the semiclassical contexts. Only, quantum mean values and certain formal quantum properties of the matter scalar field (as well as their feedback geometric responses) are replaced by their classical counterparts. In particular, the classical formulation only involves the energy density of the matter field and does not require the particle number variable, which is absent in the general relativistic formulation. The classical analog of particle production would therefore be the production of energy density of the matter field, which has to be driven (similarly to the semiclassical picture) by the self-consistently generated negative pressure associated with purely classical mechanisms. The search for such a classical cosmological scenario obviously requires a full understanding of self-consistent nonminimal scalar cosmology, and motivates the present investigation. As a bonus, it appears that the classical cosmological framework opens the way to a wider class of exact self-consistent classical solutions. Contrary to the semiclassical framework, the classical self-consistent mechanism lends itself to the consideration of arbitrary potentials (incorporating or not a cosmological constant) as well as arbitrary nonminimal couplings. This leads to a variety of exact solutions, including some that exhibit a spontaneous graceful exit from, and entry into, inflation. These solutions are ironically among those (mentioned above) usually missed in the literature. In addition, the classical analog of the semiclassical self-consistent scenario is shown and testifies to the importance of nonminimal coupling; the latter will appear to be "renormalized" according to whether the same self-consistent mechanism is realized classically or semiclassically.

These considerations are set in the framework of a general dynamical system approach to nonminimally coupled scalar field cosmology (see refs. 21 and 23 for other treatments in particular cases). The approach is very general, being valid for arbitrary scalar field potentials as well as arbitrary values of the nonminimal coupling constant ξ; the only restriction is the consideration of spatially flat Friedmann–Lemaitre–Robertson–Walker (FLRW) cosmologies.

A dimensional reduction of the differential equations governing the evolution of the system to two first-order equations leads to the result that the dynamics of a spatially flat FLRW universe fueled by a single scalar field does not admit chaos. This closes a long-standing debate [24, 25] about the possibility of chaos in the dynamics of spatially flat, homogeneous, and isotropic universes.

Further, the phase space of the self-consistent dynamical solutions has an an unexpected topological structure. The more restricted case of a massive scalar field potential with a quartic self-interaction and a cosmological constant conformally coupled to gravity is explored in a second paper [26] on exact integrability conditions of the coupled Einstein–Klein–Gordon equations.

The plan of the paper is as follows: in Section 2 we present the field equations, the conserved stress-energy tensor of the scalar field, and the dimensional reduction to a two-dimensional dynamical system, which leads to the absence of chaos. In Section 3 the topology of the phase space is investigated, together with the regions unaccessible to the orbits of the solutions and their boundary. The fixed points of the reduced dynamical system are investigated in Section 4; Section 5 discusses solutions corresponding to certain critical values of the scalar field, while Section 6 is devoted to the possibility of the universe tunneling from Minkowski space. Finally, Section 7 contains a discussion and the conclusions.

2. THE EQUATIONS AND THEIR DIMENSIONAL REDUCTION

We consider a flat FLRW spacetime with line element

$$ds^2 = d\tau^2 - a^2(\tau)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right] \tag{2.1}$$

in comoving coordinates $(\tau, r, \theta, \varphi)$, where $k = 0, \pm 1$ is the curvature index. It is assumed that the source of gravity is a scalar field ψ with a nonminimal coupling to gravity. The theory is described by the action

$$S = \frac{1}{2}\int d^4x\,\sqrt{-g}\left(-\frac{R[g]}{\kappa} + g^{\mu\nu}\psi_\mu\psi_\nu - 2V(\psi) + \xi R\psi^2\right) \tag{2.2}$$

where $F_\mu \equiv \partial F/\partial x^\mu$, $\kappa \equiv 8\pi G$, with G being Newton's constant, $V(\psi)$ is an arbitrary self-interaction potential, and ξ is the arbitrary coupling constant of the scalar field ψ to the Ricci curvature R of spacetime. The potential

$V(\psi)$ includes a possible cosmological constant as an additive term, which is written as -9Λ for economy of notations and for ease of comparison with ref. 27. We will use the dimensionless parameters

$$\alpha \equiv \kappa m^2/6, \qquad \omega \equiv \kappa^2 \Lambda \tag{2.3}$$

where m is the mass of the scalar field ψ. For example, the potential describing a massive scalar field with a quartic self-interaction in the presence of a cosmological constant (which was extensively studied in ref. 27) reads, in these notations,

$$V(\psi) = \frac{3\alpha}{\kappa}\psi^2 - \frac{\Omega}{4}\psi^4 - \frac{9\omega}{\kappa^2} \tag{2.4}$$

We shall frequently refer to this particular potential for the sake of illustration.

The notations and conventions on the metric and the Riemann and Ricci tensors are those or ref. 28 (cf. also ref. 27). In this paper the discussion is limited to spatially flat FLRW spaces, hence $k = 0$ in Eq. (2.1). The field equations derived from the action (2.2) are the Klein–Gordon equation

$$\ddot{\psi} + 3H\dot{\psi} - \xi R\psi + \frac{dV}{d\psi} = 0 \tag{2.5}$$

and the Einstein equations

$$E_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu} \tag{2.6}$$

with the Hubble function $H = \dot{a}/a$ (an overdot denotes differentiation with respect to the comoving time τ), and $T_{\mu\nu}$ is the scalar field stress-energy tensor. Let us underline that, in this approach, $T_{\mu\nu}$ refers to the expression obtained by varying the action (2.2), including the *whole* contribution due to the nonminimally coupled part. Explicitly,

$$T_{\mu\nu} = \psi_\mu\psi_\nu - \xi(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)(\psi^2) + \xi E_{\mu\nu}\psi^2 - \tfrac{1}{2}g_{\mu\nu}(\psi_\alpha\psi^\alpha - 2V) \tag{2.7}$$

where ∇_μ and $\Box$ represent, respectively, the covariant derivative and d'Alembert's operator. This approach is in contrast to the widespread attitude encountered in the literature, which consists in considering the term $\xi E_{\mu\nu}\psi^2$ as a contribution to the geometric left-hand side of the Einstein equations, thus giving rise to the "effective" Einstein equations

$$E_{\mu\nu} = \kappa_{\text{eff}}(\tau)t_{\mu\nu} \tag{2.8}$$

where

$$\kappa_{\mathrm{eff}}(\tau) = \frac{\kappa}{1 - \kappa\xi\psi^2} \tag{2.9}$$

is an effective, time-dependent, gravitational coupling. The energy density and the pressure of the scalar field are then derived from the (nonconserved) truncated energy-momentum $t_{\mu\nu}$.

This procedure is both mathematically and physically questionable or, at best, restrictive. The first reason is that the division of the Einstein equations by the factor $(1 - \kappa\xi\psi^2)$ introduces an artificial "barrier" and therefore a loss of generality. Indeed, the solutions that cross through the values $\psi = \pm 1/\sqrt{\kappa\xi}$ (for $\xi > 0$) are missed. As a consequence, the richness of the nonminimally coupled solutions is partially lost. We will indeed show that there are well-behaved dynamical solutions of the Klein–Gordon equation (2.5) and of the Einstein equations (2.6) that lie at all times on this "critical barrier." It appears, moreover, that these *critical solutions* play an unexpected dynamical role: some of them spontaneously enter into, and exit from, de Sitter regimes, even in the absence of a cosmological constant.

A supplementary reason which renders the effective reformulation of the Einstein equations suspect follows from the nonconservative character of the truncated energy-momentum tensor $t_{\mu\nu}$ in Eq. (2.8), which leads to inconsistencies such as the nonphysical negative energy densities of the scalar field. These situations are automatically avoided by the dynamical solutions of Eqs. (2.5) and (2.6). The latter can be written as the trace of the Einstein equations (2.6) and their time–time component (i.e., the Hamiltonian or energy constraint), respectively,

$$R = -6(\dot{H} + 2H^2) = -\kappa(\sigma - 3p) \tag{2.10}$$

$$3H^2 = \kappa\sigma \tag{2.11}$$

where σ and p are, respectively, the energy density and pressure of the scalar field as deduced from the energy-momentum tensor (2.7),

$$\sigma = \frac{\dot{\psi}^2}{2} + 3\xi H^2\psi^2 + 3\xi H\partial_\tau(\psi^2) + V(\psi) \tag{2.12}$$

$$p = \frac{\dot{\psi}^2}{2} - \xi[2H\partial_\tau(\psi^2) + \partial^2_{\tau\tau}(\psi^2)] - \xi(2\dot{H} + 3H^2)\psi^2 - V(\psi) \tag{2.13}$$

We stress the fact that our approach is not to prescribe *a priori* an equation of state for the scalar field ψ, but rather to determine it self-consistently together with the solutions of Eqs. (2.5) and (2.10)–(2.13). Note that the energy density σ associated with these solutions is automatically nonnegative.

The second derivative $\ddot{\psi}$ can be eliminated from the expression (2.13) of the pressure by using the Klein–Gordon equation (2.5), obtaining

$$p = \left(\frac{1}{2} - 2\xi\right)\dot{\psi}^2 + \xi H \partial_\tau(\psi^2) + 2\xi(6\xi - 1)\dot{H}\psi^2 + 3\xi(8\xi - 1)H^2\psi^2 + 2\xi\psi \frac{dV}{d\psi} - V(\psi) \tag{2.14}$$

The trace $\sigma - 3p$ of the energy-momentum tensor, which is of primary importance in the present considerations, is derived from Eqs. (2.12) and (2.13),

$$\begin{aligned}\sigma - 3p &= -\dot{\psi}^2 + 12\xi H^2\psi^2 + 9\xi H\partial_\tau(\psi^2) + 3\xi\partial^2_{\tau\tau}(\psi^2) + 6\xi\dot{H}\psi^2 + 4V(\psi) \\ &= -\dot{\psi}^2 + 9\xi H\partial_\tau(\psi^2) + 3\xi\partial^2_{\tau\tau}(\psi^2) - \xi R\psi^2 + 4V(\psi)\end{aligned} \tag{2.15}$$

or, with the help of Eq. (2.14) (the expression of the pressure modulo the Klein–Gordon equation)

$$\begin{aligned}\sigma - 3p &= \dot{\psi}^2(6\xi - 1) + 12\xi H^2\psi^2(1 - 6\xi) + 6\xi(1 - 6\xi)\dot{H}\psi^2 - 6\xi\psi\frac{dV}{d\psi} + 4V(\psi) \\ &= (6\xi - 1)(\dot{\psi}^2 + \xi R\psi^2) - 6\xi\psi\frac{dV}{d\psi} + 4V(\psi)\end{aligned} \tag{2.16}$$

In the particular case of conformal coupling $\xi = 1/6$, this leads to the simple expression

$$\sigma - 3p = 4V(\psi) - \psi\frac{dV}{d\psi} \tag{2.17}$$

which vanishes for the conformally invariant quartic self-interaction $V(\psi) = \lambda\psi^4$ [and, of course, if $V(\psi) = 0$]. This invariance property lies at the heart of the central role played by this interaction in the exact integrability properties of the dynamical equations (2.5), (2.10), and (2.11) in their conformally rescaled form; ref. 26 is devoted to this topic.

The energy conservation equation can be written in terms of these variables as

$$\dot{\sigma} + 3H(\sigma + p) = 0 \tag{2.18}$$

Thanks to the expressions (2.16) and (2.12), the Einstein equations (2.10) and (2.11) become, explicitly,

$$-6\dot{H}[1 + \xi(6\xi - 1)\kappa\psi^2] + \kappa(6\xi - 1)\dot{\psi}^2 - 12H^2 + 12\xi(1 - 6\xi)\kappa H^2\psi^2 + 4\kappa V - 6\kappa\xi\psi\frac{dV}{d\psi} = 0 \tag{2.19}$$

$$-\frac{\kappa}{2}\dot{\psi}^2 - 6\xi\kappa H\psi\dot{\psi} + 3H^2 - 3\kappa\xi H^2\psi^2 - \kappa V(\psi) = 0 \tag{2.20}$$

The dynamical equations are henceforth reduced to a closed two-dimensional first-order system for the variables H and ψ, as they only contain H, $\dot{H}$, ψ, and $\dot{\psi}$. The Hubble function H and the scalar field ψ thus appear as natural variables of the dynamical problem; this follows from the spatially flat ($k = 0$) character of the metric (2.1) considered here because the variables a and $\dot{a}$ always appear in the combination $\dot{a}/a = H$.

The dimensional reduction has an essential fallout on the unfolding of the dynamics, as the minimal dimensionality for a first-order system to exhibit chaotic behavior is three [29]. Therefore:

- *A spatially flat* ($k = 0$) *FLRW cosmological model with a scalar field source cannot exhibit chaotic behavior for arbitrary nonminimal coupling* ξ *and potential* $V(\psi)$.

Let us underline that the above-mentioned dimensional reduction is only possible in the spatially flat case. *A priori*, nothing forbids the appearance of chaotic behavior in the $k = \pm 1$ cases, and this was indeed reported in the literature [24, 30]. But then, how to understand the chaotic regimes that have also been reported in the literature [24, 25] for the spatially flat ($k = 0$) case ? It appears that these situations unavoidably correspond to *apparent* chaotic regimes due to an unexpected and strange topological structure of the phase space of the dynamical solutions.

3. TOPOLOGY OF THE PHASE SPACE

This topology is clearly exhibited by solving Eqs. (2.19) and (2.20) in terms of $\dot{H}$ and $\dot{\psi}$, respectively, which leads to the autonomous system

$$\dot{\psi} = -6\xi H\psi \pm \frac{1}{2\kappa}\sqrt{G(H, \psi)} \tag{3.1}$$

$$\dot{H} = \frac{P_1(H, \psi)}{P_2(\psi)} \tag{3.2}$$

where

$$G(H, \psi) = 8\kappa^2\left[\frac{3H^2}{\kappa} - V(\psi) + 3\xi(6\xi - 1)H^2\psi^2\right]$$
$$= 8\kappa^2[\sigma - V(\psi) + 3\xi(6\xi - 1)H^2\psi^2] \quad (3.3)$$

$$P_1(H, \psi) = 3(2\xi - 1)H^2 + 3\xi(6\xi - 1)(4\xi - 1)\kappa H^2\psi^2$$
$$\mp \xi(6\xi - 1)H\psi\sqrt{G(H, \psi)} + (1 - 2\xi)\kappa V(\psi) - \kappa\xi\psi\frac{dV}{d\psi} \quad (3.4)$$

$$P_2(\psi) = 1 + \kappa\xi(6\xi - 1)\psi^2 \quad (3.5)$$

We will see later that the function $G(H, \psi)$ plays a central role in the organization of the topological structure of the phase space of the dynamical solutions.

An inspection of the above expressions shows that some obvious dynamical restrictions are expected on the line $P_2(\psi) = 0$ for $0 < \xi < 1/6$; however, this is obviously not the case for both minimal ($\xi = 0$) and conformal ($\xi = 1/6$) couplings. This fact underlines a peculiarity of minimal and conformal couplings.

As is clear from Eqs. (3.1), (3.2), and (3.4), the vector field $(\dot{H}, \dot{\psi})$ of the system is not defined at the points where $G(H, \psi) < 0$. These points belong to regions of the (H, ψ) plane which are dynamically forbidden, i.e., unaccessible to the orbits of the solutions of the system. The boundary $\mathcal{N}$ defined in the (H, ψ) plane by

$$\mathcal{N} \equiv \{(H, \psi): G(H, \psi) = 0\} \quad (3.6)$$

may or may not exist according to the form of the potential $V(\psi)$; this curve appears as the borderline between dynamically allowed $[G(H, \psi) \geq 0]$ and forbidden $[G(H, \psi) < 0]$ regions in the (H, ψ) plane. For illustration, we show in Fig. 1 various possible situations associated with the parameters of the particular potential (2.4).

The boundary $\mathcal{N}$ induces behaviors of the solutions in the phase space which are unusual for two-dimensional dynamical systems and result in significant changes in comparison to the case in which the boundary is absent.

At this stage some considerations must be made on the reduction process which has been followed. We start from the set (2.5), (2.10), and (2.11) constituted by the Klein–Gordon equation, the trace, and the time–time component of the Einstein equations. The latter plays the role of a constraint between the variables ψ, $\dot{\psi}$, and H, while the Klein–Gordon equation and the Einstein trace equation are both of second or lower order in ψ and a. The phase space associated with this dynamical system is a three-dimensional manifold defined by the energy constraint in the four-dimensional space

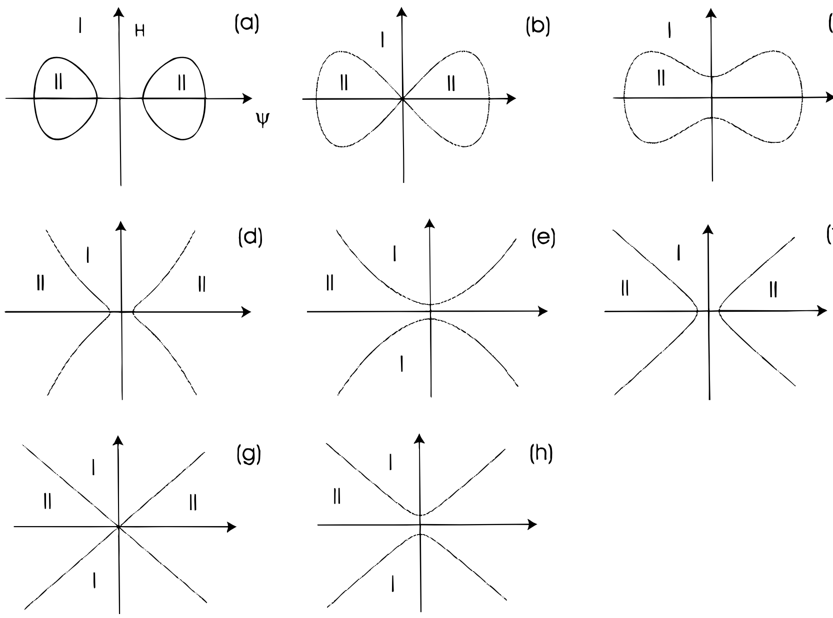

Fig. 1. Physical (I) and unphysical (II) regions in the (H, ψ) plane for $\xi = 0$, 1/6 and the potential (2.4). The vertical axis corresponds to $\psi = 0$, the horizontal axis to $H = 0$. (a) $\Omega > 0$, $\alpha^2 - \omega\Omega > 0$, $\omega > 0$. (b) $\Omega > 0$, $\omega = 0$. (c) $\Omega > 0$, $\alpha^2 - \omega\Omega > 0$, $\omega < 0$. (d) $\Omega < 0$, $\alpha^2 - \omega\Omega > 0$, $\omega \geq 0$. (e) $\Omega < 0$, $\alpha^2 - \omega\Omega > 0$, $\omega < 0$. (f) $\Omega = 0$, $\omega > 0$. (g) $\Omega = 0$, $\omega = 0$. (h) $\Omega = 0$, $\omega < 0$.

spanned by ψ, $\dot{\psi}$, a, and $\dot{a}$. By introducing the Hubble variable $H = \dot{a}/a$ we obtain a reduction to a two-dimensional system for the variables $\psi(\tau)$ and $H(\tau)$ which are therefore more "natural" variables for our model; however this result needs a careful analysis. Indeed, we solve the implicit system (2.19) and (2.20) to obtain the explicit system (3.1) and (3.2). A glance at the latter shows that it is not unique: by choosing different signs in Eqs. (3.1) and (3.2) one obtains two different systems of differential equations. What is the meaning of this nonuniqueness?

As discussed before, at any point (H, ψ) not belonging to the curve $\mathcal{N}$, whatever the chosen sign, the system (2.19) and (2.20) implies the set (3.1) and (3.2). Both systems are equivalent if both signs are considered. However, a complete set of initial conditions for the system (2.5), (2.10), and (2.11) is specified by $\psi(0)$, $\dot{\psi}(0)$, and $H(0)$, while for the system (2.19) and (2.20) one only needs the information on $\psi(0)$ and $H(0)$; one derives the value of $\dot{\psi}(0)$ from Eq. (3.1),

$$\psi_\tau(0) = -6\xi H(0)\psi(0) \pm \frac{1}{2\kappa}\sqrt{G(H(0), \psi(0))} \tag{3.7}$$

The geometric interpretation of the previous considerations is the following: the accessible phase space is made of two two-dimensional sheets joining on the boundary $\mathcal{N}$ (when the latter is present). To specify an initial condition one needs to provide the values $\psi(0)$ and $H(0)$ as well as the sign in expression (3.7), which is the same as choosing in which sheet the solution lies. The phase space topology can be said to be $R^2 \times \{+, -\}$ with holes corresponding to the regions where $G < 0$. In other words, the solution (H, ψ) of Eqs. (3.1) and (3.2) is the projection on the (H, ψ) plane of the original solution living on the two-dimensional zero-energy submanifold of the original phase space $(H, \psi, \dot{\psi})$. This solution is recovered by lifting (positive sign) or lowering (negative sign) the projection using Eqs. (3.1) and (3.2) at each point of the (H, ψ) plane. The two possible signs in Eq. (3.1) are related to the reversibility of the physical processes described by such equations. Reversibility is guaranteed in Eqs. (3.1) and (3.2) in the following way: if $(H(\tau), \psi(\tau))$ is a solution of the system (3.1) and (3.2) with the positive (negative) sign, then $(-H(-\tau), \psi(-\tau))$ is a solution of the same system with the reversed sign.

This rather involved phase-space structure may be the cause of the apparent chaotic regimes that have been reported for $k = 0$ (flat spatial sections) [25]. Indeed, numerical integration cannot exactly distinguish points belonging to the allowed region from points located in the "nearby" forbidden region. Furthermore, the discretization of the dynamics generally tends to worsen these discrepancies; it is therefore expected that these pseudochaotic regimes manifest themselves in neighborhoods of the boundary $\mathcal{N}$. This issue is presently under investigation and will be reported upon in a forthcoming publication.

In order comprehensively to explore the topological organization of the orbits of the dynamical solutions and the central importance of the function $G(H, \psi)$, one should keep in mind that the dynamically available phase space is a two-dimensional surface in the original three-dimensional space $(H, \psi, \dot{\psi})$. The topology of this surface depends on the potential $V(\psi)$ and (crucially) on the existence of the boundary $\mathcal{N}$.

The surface is connected without holes (i.e., dynamically forbidden regions) when G is nonnegative, that is, when the boundary $\mathcal{N}$ does not exist. On the contrary, when the potential $V(\psi)$ is such that G has negative sectors, the surface exhibits holes or may even be nonconnected (see Fig. 1 for examples). Equations (3.1) and (3.2), along with the choice of the positive or negative sign, yield a complete description of the orbits on that surface from their projection in the (H, ψ) plane. These projected orbits are, of course, restricted by the possible presence of negativity regions for G, bounded by the curve $\mathcal{N}$. Indeed, these orbits are entirely confined to the positivity region of the function G and therefore never cross the boundary $\mathcal{N}$ toward a dynamically forbidden region $G < 0$ in the (H, ψ) plane. An important question is the following: does the boundary $\mathcal{N}$ itself belong to the forbidden or to the allowed region or, in other words, are the orbits allowed to touch $\mathcal{N}$? A mathematical rephrasing of the question is: does the expression given by Eq. (3.1) satisfy the Klein–Gordon equation (2.5) on the $G = 0$ boundary? The solution of the problem depends on the behavior of the function $G(H, \psi)$ in a neighborhood of $G = 0$, hence on $\dot{G}$ evaluated at $G = 0$, where

$$\dot{G} = \frac{\partial G}{\partial H}\dot{H} + \frac{\partial G}{\partial \psi}\dot{\psi} \tag{3.8}$$

By combining Eqs. (3.8) and (3.1)–(3.3) one obtains

$$\dot{G} = \pm 24\xi(1 - 6\xi)\kappa H^2\psi\sqrt{G} - 12\left(\frac{1}{2} - \xi\right)HG \mp 4\kappa\sqrt{G}\,\frac{dV}{d\psi} \tag{3.9}$$

It follows from Eq. (3.9) that

$$\dot{G}(H, \psi) = 0 \text{ on the boundary } \mathcal{N} \tag{3.10}$$

and that $\dot{G}/\sqrt{G}$ is well behaved on $\mathcal{N}$,

$$\left.\frac{\dot{G}}{\sqrt{G}}\right|_{G=0} = \pm 24\kappa\xi(1 - 6\xi)H^2\psi \mp 4\kappa\,\frac{dV}{d\psi} \tag{3.11}$$

From Eq. (3.1) one deduces that

$$\ddot{\psi} = -6\xi\partial_\tau(H\psi) \pm \frac{1}{4\kappa}\frac{\dot{G}}{\sqrt{G}} \tag{3.12}$$

Then it immediately follows from Eqs. (3.1), (3.2), (3.11), and (3.12) that the Klein– Gordon equation (2.5) is satisfied on the boundary $\mathcal{N}$. The latter then belongs to the allowed region and consequently may be reached by the orbits of the dynamical solutions. Once this happens, the orbits of such solutions are repelled toward the allowed positivity region of $G(H, \psi)$ (except for the case in which the contact point is a fixed point, as clarified below). In other words, these solutions cannot be trapped by the boundary $\mathcal{N}$ and propagate on it, as explained below.

The orbits of these "bouncing" solutions on $\mathcal{N}$, reconstructed from their projections on the (H, ψ) plane into the original two-dimensional phase space surface, originate from one sheet, touch the curve $\mathcal{N}$, and afterward evolve on the other sheet of the two-dimensional surface. This explains the possible apparent pathology (i.e., the nonuniqueness of the solutions) of the phase portrait in the neighborhood of $\mathcal{N}$, since some projected orbits may then cross each other. However, these crossing orbits are only projections of (noncrossing) curves transiting from one sheet to the other.

The constraint $G = 0$ imposed *along* the orbit of a dynamical solution is only compatible with Eqs. (2.5), (3.1), and (3.2) if this solution is a fixed point of the dynamical system, namely

$$\dot{\psi} = 0 \tag{3.13}$$

$$\dot{H} = 0 \tag{3.14}$$

Hence, only fixed points of the dynamical system may possibly live on the boundary $\mathcal{N}$; but this is a necessary and sufficient condition only for minimal coupling. Indeed it follows from Eqs. (3.1) and (3.2)–(3.5) that, for $\xi = 0$,

$$\dot{\psi} = \pm\frac{1}{2\kappa}\sqrt{G} \tag{3.15}$$

$$\dot{H} = -\frac{G}{8\kappa} \tag{3.16}$$

and *all* the fixed points are located on the boundary $\mathcal{N}$ in this case. *This is not the case for arbitrary nonminimal coupling*, as we shall show now.

4. FIXED POINTS AND OTHER SOLUTIONS

Let us denote a fixed point, solution of Eqs. (3.13) and (3.14), by (H_0, ψ_0). The Klein–Gordon equation (2.5) then leads to

$$12\xi H_0^2\psi_0 + V_0' = 0 \tag{4.1}$$

where $V_0' \equiv dV/d\psi|_{\psi_0}$.

(i) If $\psi_0 = 0$ or $\xi = 0$ (or both), then Eq. (4.1) implies $V_0' = 0$. The corresponding value of H_0 is then deduced from Eqs. (2.11) and (2.12), which lead to

$$H_0^2 = \kappa V(\phi_0)/3 \tag{4.2}$$

This fixed point obviously exists only if $V(\psi_0) \geq 0$. In the particular case of the example potential (2.4) the fixed point $(\pm\sqrt{\kappa V(\psi_0)/3}, 0)$ reduces to the *purely geometric de Sitter* fixed point (as defined in ref. 27), due only to the presence of a cosmological constant.

(ii) If $\psi_0 \neq 0$ (and $\xi \neq 0$), then it follows from Eq. (4.1) that

$$H_0^2 = -\frac{1}{12\xi\psi_0} V_0' \tag{4.3}$$

which requires the positivity of the right-hand side. The corresponding value of ψ_0 is then deduced from either the energy equation (2.11) or the trace equation (2.10); note that these equations are identical in this case. Indeed, it follows from both of them that

$$\dot{H} = -\frac{\kappa}{2}(\sigma + p) \tag{4.4}$$

and consequently the de Sitter equation of state

$$\sigma + p = 0 \tag{4.5}$$

follows from the condition $\dot{H} = 0$. This implies, in turn, that the trace equation (2.10) becomes

$$R = -6(\dot{H} + 2H^2) = -12H^2 = -\kappa(\sigma - 3p) = -4\kappa\sigma \tag{4.6}$$

thus yielding

$$\kappa\sigma = 3H^2 \tag{4.7}$$

which is precisely the energy equation (2.11). The latter takes the form

$$\sigma(\psi_0) = 3\xi H_0^2\psi_0^2 + V(\psi_0) = 3H_0^2/\kappa \tag{4.8}$$

By combining Eqs. (4.8) and (4.3), one then obtains

$$4\kappa\xi\psi_0 V(\psi_0) + V_0'(1 - \kappa\xi\psi_0^2) = 0 \tag{4.9}$$

The roots of this algebraic equation are the fixed-point values of ψ_0 associated with the potential $V(\psi)$ and the nonminimal coupling constant ξ.

In the particular case of the potential (2.4), Eqs. (4.3) and (4.9) lead to

$$H^2 = \frac{3(\alpha^2 - \Omega\omega)}{\kappa(\Omega - 6\xi\alpha)} \tag{4.10}$$

$$\psi^2 = \frac{6(\alpha - 6\xi\omega)}{\kappa(\Omega - 6\xi\alpha)} \tag{4.11}$$

provided that $\Omega \neq 6\alpha\xi$. The singular situation $\Omega = 6\alpha\xi$ (discussed in ref. 27) corresponds precisely to the de Sitter integrability condition $\alpha = \omega = \Omega$ and $\xi = 1/6$ of ref. 26.

The expressions (4.10) and (4.11) allow one to answer the question of whether a de Sitter self-consistent classical solution can reproduce the semiclassical de Sitter regime. The latter was obtained [11] for a massive quantum scalar field conformally coupled to gravitation without any additional self-interaction or cosmological constant. Here, this situation corresponds to $\alpha \neq 0$ and $\omega = \Omega = 0$, which leads to

$$H^2 = -\frac{\alpha}{2\kappa\xi} \tag{4.12}$$

$$\psi^2 = -\frac{1}{\kappa\xi} \tag{4.13}$$

Obviously H and ψ only exist if $\xi < 0$. Hence, the classical solution mimicking the semiclassical one requires a shift of the nonminimal coupling constant ξ from its semiclassical conformal value 1/6 to $\xi < 0$.

One recovers the late-time mild inflationary scenario of ref. 20 which corresponds to short periods of exponential expansion of the universe interrupting the present matter-dominated era. The scenario was introduced to overcome the age of the universe problem and the problem of the discrepancy between locally and globally measured values of the Hubble parameter [20]. Late-time mild inflation was achieved by assuming that the dark matter is dominated by a nonminimally coupled scalar field with $\Omega = \omega = 0$, $\alpha > 0$, $\xi < 0$, and $|\xi| \gg 1$ (strong coupling), and corresponds to the solution [20]

$$(H, \psi) = \left(\sqrt{\frac{\alpha}{2|\xi|\kappa}}, \pm\frac{1}{\sqrt{\kappa|\xi|}}\right) \tag{4.14}$$

This solution is obtained from Eqs. (4.10) and (4.11) for $\omega = \Omega = 0$ and $\xi < 0$ and is *unstable*, a feature that is desirable in order to stop the exponential expansion of the universe soon after it starts [20]. When $|\xi| \gg 1$, the spacetime given by this solution exhibits pathologies that make it physically unacceptable, as explained in detail in ref. 31. The latter paper connects the physics of a nonminimally coupled scalar field to the study of tails of scalar waves in curved spaces, which can have significant implications for cosmology.

By combining Eqs. (4.3) and (4.9) one obtains the corresponding value of H_0,

$$H_0^2 = \frac{\kappa V(\psi_0)}{3(1 - \kappa\xi\psi_0^2)} \tag{4.15}$$

together with the condition

$$\kappa\xi\psi_0^2 \neq 1 \tag{4.16}$$

The case

$$\kappa\xi\psi_0^2 = 1, \qquad \psi_0 = \psi_c \equiv \pm\frac{1}{\sqrt{\kappa\xi}} \tag{4.17}$$

where ψ_c denotes the *critical value of the field* ψ (for positive values of ξ), will later appear to play a crucial role in the unfolding of a self-consistent dynamical behavior by endowing the latter with unexpected "critical field solution." Before discussing this issue, we note that there might still exist a "usual" fixed point associated with the critical field value ψ_c. It follows from Eq. (4.9) that

$$V_c \equiv V(\psi_c) = 0 \tag{4.18}$$

The critical field value ψ_c is therefore a fixed point if the potential vanishes there, and the corresponding value of H_0 is derived from Eq. (4.3),

$$H_0^2 = -\frac{1}{12\xi\psi_c} V_c' = \mp\frac{1}{12}\sqrt{\frac{\kappa}{\xi}}\, V_c' \tag{4.19}$$

where the $\pm$ signs refer to the two values of ψ_c in Eq. (4.17).

Let us now come back to the problem previously raised on the possible location of the fixed points on the boundary $\mathcal{N}$. Precisely, what are the conditions for a fixed point to be located on $\mathcal{N}$ in the case of arbitrary nonminimal coupling ξ? These conditions are obtained by combining the relations (4.1) and (4.9) for the fixed points (H_0, ψ_0) with the equation for the boundary $\mathcal{N}$; this leads to the simple constraint

$$\psi_0 V_0' = 0 \tag{4.20}$$

this is satisfied, e.g., if $\psi_0 \neq 0$ and the potential $V(\psi)$ is stationary at that point, i.e.,

$$\psi_0 \neq 0, \qquad V_0' = 0 \tag{4.21}$$

But Eq. (4.21) combined with Eq. (4.9) implies that

$$V(\psi_0) = 0 \tag{4.22}$$

Moreover, the value of H_0 given by the Klein–Gordon equation (4.1) is

$$H_0 = 0 \tag{4.23}$$

This result also follows from the conditions $G = 0$ and $V(\psi_0) = 0$. In the terminology of ref. 27 this fixed point is therefore a *nontrivial Minkowski space* ("nontrivial" because it is realized with a nontrivial potential $V(\psi)$ and a nonvanishing value ψ_0 of the scalar field). Referring once more to the example case of the potential (2.4), this situation is realized when the three parameters α, Ω, and ω satisfy the constraint $\alpha^2 = \omega\Omega$, and the corresponding fixed point is then given by

$$(H_0, \psi_0) = \left(0, \pm\sqrt{\frac{6\alpha}{\kappa\Omega}}\right) \tag{4.24}$$

Another realization of the constraint (4.20) is obtained with

$$\psi_0 = 0 \tag{4.25}$$

and the only information delivered by the Klein–Gordon equation is then $V'_0 = 0$. In this case the value of H_0 is directly deduced from Eqs. (2.11) and (2.12),

$$H_0^2 = \frac{\kappa}{3} V(0) \tag{4.26}$$

with the condition $V(0) \geq 0$. This fixed point

$$(H_0, \psi_0) = \left(\pm \sqrt{\frac{\kappa V(0)}{3}}, 0 \right) \tag{4.27}$$

represents, once more in the terminology of ref. 27, a *geometric de Sitter* fixed point. This is realized, e.g., when the potential reduces to a pure cosmological constant, as in ref. 27.

In the preceding discussion concerning the fixed points of the dynamical system, we saw the emergence of special values of the scalar field [see Eqs. (4.9) and (4.15)], namely the critical values ψ_c given by Eq. (4.17). Why do such special values appear in the discussion, and what is their physical and mathematical meaning? It follows from Eqs. (2.12) and (2.13) that the energy density σ and pressure p associated to a constant scalar field ψ_0 obey the relation

$$\sigma + p = -2\xi\psi_0^2\dot{H} \tag{4.28}$$

By combining this equation with Eq. (4.4) one obtains

$$\dot{H}(1 - \kappa\xi\psi_0^2) = 0 \tag{4.29}$$

from which one deduces that $\dot{H} = 0$ when $\kappa\xi\psi_0^2 \neq 1$ (i.e., $\psi \neq \psi_c$); this in turn yields the fixed points already discussed. On the contrary, the special case $\psi = \psi_c$ leads to two possibilities: the first one corresponds to the simultaneous vanishing of the two terms in the left-hand side of Eq. (4.29) and corresponds to the fixed point (4.19) located at the critical value of the scalar field. The second realization of the constraint (4.29) corresponds to the unexpected dynamical situation in which $\dot{H} \neq 0$, hence to a time-dependent Hubble function $H(\tau)$ associated with a constant value ψ_c of the scalar field. How is this possibility, offerred *a priori* by the constraint (4.29), realized dynamically? In other words, does the set of dynamical equations (2.5), (2.10), and (2.11) offer such an unusual solution, and under which conditions?

The Klein–Gordon equation (2.5) leads to

$$-\xi R\psi_c + V'_c = 0 \tag{4.30}$$

where $V'_c \equiv dV/d\psi|_{\psi_c}$, and

$$R = -6(\dot{H} + 2H^2) = \frac{1}{\xi\psi_c} V'_c \tag{4.31}$$

is therefore a constant. The time-dependent Hubble function $H(\tau)$ satisfies a Riccati equation characterizing this constant-curvature space,

$$\dot{H} + 2H^2 = -\frac{1}{6\xi\psi_c} V'_c = -\frac{R}{6} \tag{4.32}$$

The trace equation (2.10) has then to be satisfied together with the expression of the trace (2.15), which reduces to

$$\sigma - 3p = -\xi R\psi_c^2 + 4V_c \tag{4.33}$$

Hence, following Eq. (2.10),

$$R = -\kappa(\sigma - 3p) = \kappa\xi\psi_c^2 R - 4\kappa V_c \tag{4.34}$$

Since $\kappa\xi\psi_c^2 = 1$ by definition, this equation is obviously satisfied only if

$$V_c = 0 \tag{4.35}$$

This condition will be discussed later.

Finally, the energy density derived from from Eq. (2.12) is, in this case,

$$\sigma = 3\xi\psi_c^2 H^2 + V(\psi_c) \tag{4.36}$$

Equations (4.36), (4.17), and (4.35) yield

$$\sigma = 3H^2/\kappa \tag{4.37}$$

which coincides with the energy constraint (2.11). A last, but crucial, point to be checked is whether these critical $(H(\tau), \psi_c)$ are physically dynamical solutions and belong to the allowed region of the (H, ψ) phase space for any potential $V(\psi)$ and nonminimal coupling ξ. The function $G(H, \psi)$ given by Eq. (3.3), when combined with Eqs. (4.17) and (4.35), becomes

$$G(H, \psi_c) = 144\xi\kappa H^2 \tag{4.38}$$

which is nonnegative for any positive ξ (the entire range of values of ξ for which the critical value ψ_c is defined). The critical solutions $H(\tau)$ are therefore entirely confined to the allowed dynamical region. Moreover, following Eq. (4.38), these solutions are only entitled to "touch" the boundary $\mathcal{N}$ at the points where $H(\tau) = 0$, hence on the ψ axis in the (H, ψ) plane. The orbits of the critical solutions are lines parallel to the H axis and crossing the ψ axis at the value $\psi = \psi_c$.

Before discussing explicitly the behavior of these critical solutions, as deduced from Eq. (4.32), we emphasize several of their generic aspects. The first one refers to the *universality* of these solutions, in the following sense: there are only three critical solutions $H(\tau)$, according to whether the right-hand side of Eq. (4.32) is positive, zero, or negative. This depends, in turn, only on the sign of the expression V_c'/ψ_c, without reference to other features of the potential $V(\psi)$, and for any positive value of the nonminimal coupling constant. In addition, the two specifications of the critical values ψ_c given by (4.17) and (4.35) amplify their universality character for the following reasons:

(i) If a given potential $V(\psi)$ vanishes at some value ψ_0 of the scalar field ψ, one may *adjust* the nonminimal coupling constant ξ to the value $\xi_0 = (\kappa\psi_0^2)^{-1}$. This promotes the value ψ_0 to the critical one ψ_c associated with this potential $V(\psi)$ in the presence of the nonminimal coupling ξ_0.

(ii) If a given potential $V(\psi)$ does not possess any zeros, one may "shift" $V(\psi)$ by adding a constant term (a cosmological constant) such that the newly defined potential vanishes at a prescribed value ψ_0. The latter is then promoted to the critical value associated with this potential by adjusting the nonminimal coupling parameter ξ to the value $\xi_0 = (\kappa\psi_0^2)^{-1}$, as described in (i).

There are situations in theoretical physics and in cosmology in which ξ is a running coupling. The study of asymptotically free theories in an external gravitational field shows a scale-dependent coupling parameter $\xi(\epsilon)$. In refs. 32 and 33 it was shown that asymptotically free GUTs have a ξ depending on a renormalization group parameter ϵ, and that $\xi(\epsilon)$ converges to 1/6, ∞, or to any initial condition ξ_0 as $\epsilon \rightarrow +\infty$ (this limit corresponds to strong curvature conditions and to the early universe), depending on the gauge group and on the matter content of the theory. In ref. 34 it was also obtained that $|\xi(\epsilon)| \rightarrow +\infty$ in $SU(5)$ GUTs. Similar results were derived in finite GUTs, with the convergence of ξ to its asymptotic value being much faster [32, 33]. An exact renormalization group study of the $\lambda\phi^4$ theory shows that $\xi = 1/6$ is a stable infrared fixed point [35]. The running of the coupling ξ was employed in cosmology before [36, 18, 19].

We conjecture that this universality confers on these self-consistent critical solutions a universal physical role in the unfolding of cosmological nonminimal, self-consistent scalar dynamics beyond the peculiarities of the Lagrangian in the action (2.2). As an example, consider the exact solution (see Section 5 for its derivation)

$$a = a_0 \cosh^{1/2}[\sqrt{2C}(\tau - \tau_0)], \qquad \psi = \psi_c \tag{4.39}$$

$$H = \sqrt{\frac{C}{2}} \tanh[\sqrt{2C}(\tau - \tau_0)] \tag{4.40}$$

which is of special physical interest since it describes a nonsingular cosmology which spontaneously emerges from a contracting asymptotic ($\tau \rightarrow -\infty$) de

Sitter regime, contracts to a minimum nonvanishing size, and then gracefully reenters into an asymptotic expanding de Sitter regime (for $\tau \to +\infty$). The presence or absence of a cosmological constant in order to achieve this dynamical behavior is unimportant.

In view of the universality of this behavior (in the sense described above), we emphasize the fact that this dynamical evolution is inherent to a wide class of potentials and nonminimal couplings. Ironically, these critical solutions are precisely the ones which are often missed or at least ignored in the literature due to the misleading and restrictive use of the effective gravitational constant given by Eq. (2.9). On the contrary, not only does our self-consistent approach include these critical field solutions, it also raises them to an unexpected status.

It might erroneously be deduced from the considerations about the fixed points that $\psi = \text{const}$ is a necessary condition for a dynamical solution to be a de Sitter one. Indeed, by contrast with the minimally coupled case ($\xi = 0$) in which this property holds, the nonminimal case admits nontrivial dynamical realizations of the fixed-point constraints [37]

$$H = \text{const} \tag{4.41}$$

$$\psi = \psi(\tau) \tag{4.42}$$

According to Eqs. (3.1)–(3.3), this corresponds to

$$\dot{\psi} = -6\xi H\psi \pm \sqrt{G(H, \psi; \xi; V(\psi))} \tag{4.43}$$

$$P_1(H, \psi; \xi; V(\psi)) = 0 \tag{4.44}$$

5. CRITICAL FIELD SOLUTIONS

These solutions correspond to constant Ricci curvature. Since $R = -6(\dot{H} + 2H^2)$ in a spatially flat FLRW universe, the condition $R = \text{const}$ is equivalent to

$$\dot{H} + 2H^2 = C \tag{5.1}$$

where $C = -R/6$ is a constant. Equation (2.10) then yields

$$p = \frac{\sigma}{3} + p_0 = \frac{\sigma}{3} + \left(\frac{R}{3\kappa}\right) \tag{5.2}$$

One could continue by noticing that Eq. (5.1) is a Riccati equation, and by applying the standard methods for its solutions, but we prefer to proceed as follows.

For $C = 0$, Eq. (5.1) is immediately integrated to obtain the Minkowski space corresponding to $H = 0$ or the radiation solution

$$H = \frac{1}{2(\tau - \tau_0)} \tag{5.3}$$

where τ_0 is an integration constant, and

$$a = a_0\sqrt{\tau - \tau_0} \tag{5.4}$$

For $C > 0$ one has

$$\frac{\dot{H}}{1 - 2H^2/C} = C \tag{5.5}$$

which leads to

$$\int \frac{dx}{1 - x^2} = \sqrt{2C}(\tau - \tau_0) \tag{5.6}$$

where $x \equiv \sqrt{2/C}H$. Since the value of the integral in Eq. (5.6) is arctgh x if $x^2 > 1$ and $\ln[((1 + x)/(1 - x))^{1/2}]$ if $x^2 < 1$ one obtains, after elementary calculations,

$$H = \sqrt{\frac{C}{2}}\tanh[\sqrt{2C}(\tau - \tau_0)] \tag{5.7}$$

for both $|H| > \sqrt{C/2}$ and $|H| < \sqrt{C/2}$. Equation (5.7) is incompatible with the first limit on $|H|$ and therefore there is no solution for $|H| > \sqrt{C/2}$. The scale factor corresponding to the solution for $|H| < \sqrt{C/2}$ is

$$a = a_0 \cosh^{1/2}[\sqrt{2C}(\tau - \tau_0)] \tag{5.8}$$

representing a nonsingular universe which contracts from an asymptotic pure de Sitter regime ($\tau \to -\infty$), bounces at a minimum size a_0 ($\tau = \tau_0$), and reexpands to a pure de Sitter regime ($\tau \to +\infty$).

The cases $H = \pm\sqrt{C/2}$ not included in Eq. (5.5) correspond to the de Sitter solutions

$$a = a_0 \exp\left[\pm\sqrt{\frac{C}{2}}(\tau - \tau_0)\right] \qquad \left(H = \pm\sqrt{\frac{C}{2}}\right) \tag{5.9}$$

The solution (5.9) with $\psi = \pm\psi_c$ corresponds to the vanishing of *both* $\dot{H}$ and $(1 - \kappa\xi\psi^2)$ in Eq. (4.29).

For $C < 0$ one reduces Eq. (5.1) to

$$\int \frac{dx}{1 + x^2} = -\sqrt{2|C|}(\tau - \tau_0) \tag{5.10}$$

(where $x \equiv \sqrt{2/|C|}H$), which is immediately integrated,

$$H = -\sqrt{\frac{|C|}{2}}\tan[\sqrt{2|C|}(\tau - \tau_0)] \tag{5.11}$$

and finally

$$a = a_0 \cos^{1/2}[\sqrt{2|C|}(\tau - \tau_0)] \tag{5.12}$$

which represents a universe expanding from a big bang singularity to a maximum size and then recollapsing.

The structure of the equation of state (5.2) is remarkable as its "quasi-radiation-like" form hides a wide class of behaviors, which are apparent when one rewrites Eq. (5.2) with the expression (2.10) of R,

$$p = \frac{\sigma}{3} - 2\frac{\dot{H}}{\kappa} - 4\frac{H^2}{\kappa} = \frac{\sigma}{3} - 4\frac{\sigma}{3} - 2\frac{\dot{H}}{\kappa} = -\sigma - 2\frac{\dot{H}}{\kappa} \tag{5.13}$$

The equation of state is then written in a "quasi-de Sitter-like" form, and it is clear how it may continuously evolve to a pure de Sitter one, reached when $\dot{H} = 0$. This property of the critical field solutions lies behind the fact that they can spontaneously enter or exit an inflationary de Sitter regime. Of the three solutions, this property is enjoyed by (5.7), which exhibits a peculiarity: the (time-dependent) pressure associated with its evolution is *permanently negative*:

$$\sigma + p \leq 0 \tag{5.14}$$

This feature follows from Eq. (5.13) and from the fact that, according to Eq. (5.7),

$$\dot{H} \geq 0 \tag{5.15}$$

The equality in Eq. (5.14) (hence the vacuum equation of state and the de Sitter regime) is reached when $\dot{H} = 0$, which happens as $\tau \to \pm\infty$, when $H(\tau)$ reaches its extremal values

$$H_C = \pm\sqrt{\frac{C}{2}} = \pm\sqrt{\frac{|R|}{12}} = \pm\sqrt{\frac{|V_c'|}{12\xi\psi_c}} \tag{5.16}$$

In the purely classical context, this negative pressure drives the creation of the scalar field energy density σ. Since this solution is simultaneously characterized by $V_c = 0$ [see Eq. (4.35)] and $\psi = \text{const} = \psi_c$, the energy density σ receives no contributions from the kinetic energy density of the scalar field, nor from the potential energy density $V(\psi)$. This fact naturally lends itself to the interpretation of the variation of σ as being solely due to the change in the number of "quanta" of the scalar field ψ, independently of the precise definition of quanta. Hence, this solution represents a purely

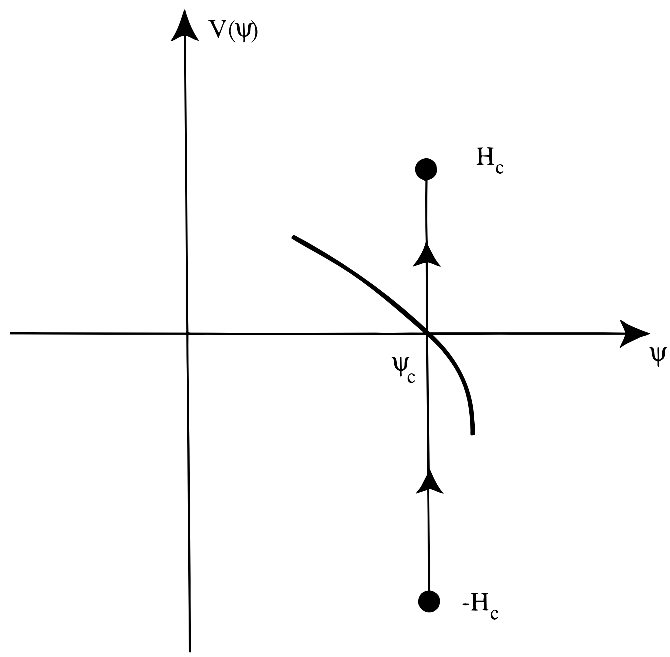

Fig. 2. The general shape of the potentional $V(\psi)$ which engenders the critical solution (5.7). The requirements are $V(\psi_c) = 0$ and $dV/d\psi|_{\psi_c} \equiv V_c^l < 0$.

classical analog of the semiclassical self-consistent cooperative mechanism [11] in which the spacetime evolution involves scalar particle production (or annihilation), and their feedback response is precisely the one required to sustain the evolution.

As we already mentioned, this solution spontaneously passes from an asymptotic ($\tau \to -\infty$) contracting de Sitter regime to an asymptotic ($\tau \to +\infty$) expanding de Sitter regime with an intermediate ($\tau = \tau_0$) bounce of the scale factor corresponding to $H(\tau)$ passing through the value $H = 0$. According to Eq. (4.38), this happens when the solution (5.7) touches the boundary $\mathcal{N}$. It follows from Eqs. (4.18), (4.31) and (5.1) that the general behavior of the potential $V(\psi)$ which engenders the critical solution (5.7) is the one described by Fig. 2. We saw that the orbit of this solution, which is a straight line parallel to the H axis and crossing the ψ axis at the point $(0, \psi_c)$, touches the $G = 0$ boundary precisely at that point. According to Eq. (3.3), the forbidden region $G < 0$ necessarily corresponds to values $\psi < \psi_c$ [where $V(\psi) > V_c$]. The boundary $\mathcal{N}$, being described by an even function of H, appears therefore as in Fig. 3.

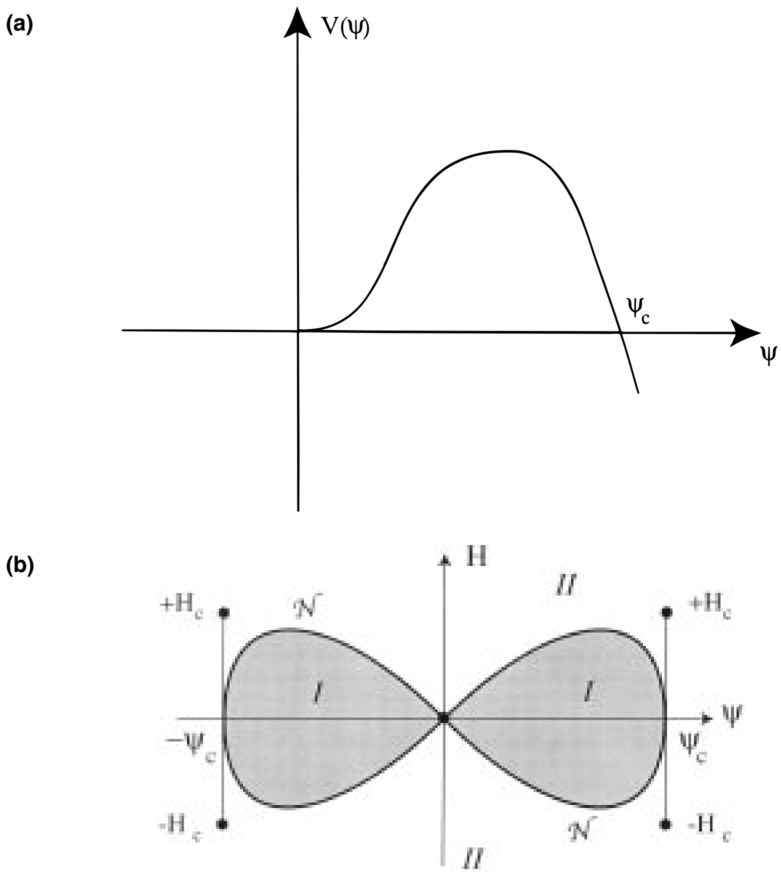

Fig. 3. (a) The general shape of the potential $V(\psi)$ which produces the configuration depicted in panel b. (b) The forbidden dynamical region (I), the allowed region (II), the boundary $\mathcal{N}$, and the orbits of the critical solutions (5.7). The trivial Minkowski fixed point is located at (0, 0); the four de Sitter fixed points are $(\pm H_c, \pm\psi_c)$.

One can illustrate the situation with the help of the potential (2.4) in the case of conformal coupling $\xi = 1/6$: the critical solutions $\psi_c = \pm\sqrt{6/\kappa}$ are realized with this potential when $\Omega = 2\alpha$ and $\omega = 0$, i.e.,

$$V(\psi) = \frac{3\alpha}{\kappa}\psi^2 - \frac{\alpha}{2}\psi^4 \tag{5.17}$$

The function $G(H, \psi)$ in Eq. (3.3) is

$$G(H, \psi) = 8\kappa^2\left[\frac{3H^2}{\kappa} - \frac{3\alpha}{\kappa}\psi^2 + \frac{\alpha}{2}\psi^4\right] \tag{5.18}$$

There are five fixed points (H_0, ψ_0): one located at the origin and the four de Sitter ones $(\pm\sqrt{3\alpha/\kappa}, \pm\sqrt{6/\kappa})$, the endpoints of the orbits of the two symmetric critical solutions $\pm\psi_c$, respectively. The values of $\pm H_c$ are deduced from Eqs. (4.32) and (5.18), or from Eq. (4.10) with $\Omega = 2\alpha$, $\omega = 0$, and $\xi = 1/6$. We stress that the configuration of the critical solutions $\pm H_c(\tau)$, the boundary $\mathcal{N}$, the forbidden region, as well as the relative location of the fixed points, are valid for any potential with the shape shown in Fig. 3, hence independently of the particular form (5.17) given for the sake of illustration. Only, the value of $H_c^2 = -\frac{1}{2}V_c'/\psi_c = -\frac{1}{2}\sqrt{\kappa/6}V_c'$ depends on V_c'. Therefore, the discussion of the next section is valid for this entire class of potentials.

As far as the two remaining critical field solutions (5.3) and (5.11) are concerned, we show that, in contrast to the one discussed above, they are characterized by positive pressure. Since they satisfy the hypotheses of the Hawking–Penrose singularity theorems, they both exhibit a big bang singularity. The solution (5.3) corresponds to vanishing Ricci curvature $R = 0$ and, according to Eq. (5.2), is a pure radiation solution. It follows from Eqs. (4.22) and (4.31) that this solution is realized, among others, by a *free* [i.e., $V(\psi) = 0$] nonminimally coupled scalar field. Moreover, according to the previous discussion on the universality of the critical solutions, one may adjust the nonminimal coupling constant ξ in such a way that any prescribed value ξ_0 becomes the critical value; in this case the coupling constant ξ is the only adjustable parameter.

The solution (5.11) is associated with a positive value of the Ricci curvature and represents a universe starting from a big bang singularity, reaching a maximum size, and then recollapsing on the boundary $\mathcal{N}$.

6. WILD IDEAS

The topological organization of the dynamically forbidden and allowed regions, the shape of the boundary $\mathcal{N}$, and the location of the fixed points associated with the potential (5.17) allow for the intriguing possibility of a semiclassical "birth of the universe from the Minkowski vacuum." To avoid confusion, let us stress that this proposal is distinct from the "birth of the universe from nothing" advocated in quantum cosmology [38]. In our case the "primordial cosmological state" does not refer to a state without classical space and time, as in quantum cosmology, but rather to the Minkowski space (unstable) fixed point $(H, \psi) = (0, 0)$ located at the origin of the (H, ψ) phase space. Hence, our mechanism does not involve quantum tunneling of the scale factor, but merely the quantum vacuum fluctuations of the scalar field in the Minkowski classical spacetime background. The scalar field is thereby promoted to the role of a quantum field in a semiclassical context. The quantum fluctuations of the scalar field which reach the critical amplitude ψ_c may be viewed as giving rise to a tunneling in the phase space through the classically forbidden region until it reaches the nearest point allowed by classical dynamics on the ψ axis, namely $(0, \psi_c)$. By so doing, the cosmological system (H, ψ) is carried from an unstable Minkowski fixed point $(0, 0)$ through a classically forbidden region, toward the classically allowed point $(0, \psi_c)$. The latter plays the role of initial condition for the subsequent cosmological evolution, i.e., the *unique* classical solution emanating from this point, the critical field solution (5.7). The classical point $(0, \psi_c)$ has a special cosmological status in this picture: the Hubble function H and the energy density σ of the scalar field vanish there, as well as in their original (trivial) Minkowski space configuration before tunneling. On the contrary, the time derivative $\dot{H}$ acquires a positive value deduced from Eq. (5.1),

$$\dot{H} = -\sqrt{\frac{\kappa}{6}}\, V_c' = \frac{6\alpha}{\kappa} > 0 \tag{6.1}$$

The pressure given by Eq. (5.13) is

$$p = -\sigma - \frac{12\alpha}{\kappa^2} = -12\,\frac{\alpha}{\kappa^2} < 0 \tag{6.2}$$

This negative pressure drives the cosmological system out of the ψ axis, along the orbit of the critical field solution, and owes its existence to the nonminimal coupling term $-2\xi\dot{H}\psi_c^2$ in Eq. (2.13).

As a summary, the jump from the primordial vacuum configuration $(H, \psi) = (0, 0)$ to the nonvacuum one $(H, \psi) = (0, \psi_c)$ manifests itself in a *finite* discontinuity of the pressure

$$\Delta p = -12\alpha/\kappa^2 \tag{6.3}$$

or, in other words, in a "push" ($H = 0$ and $\dot{H} > 0$) on the time derivative $\dot{H}$ of the Hubble function, which acquires the value (6.1). The latter, which results from the backreaction of the classical source ψ_c, expresses the acceleration of the cosmological system along the orbit of its critical field solution; the point representing the universe in the phase space continuously slows down along its orbit (in the sense that $\dot{H}$ decreases), until its "acceleration" $\dot{H}$ vanishes approaching the de Sitter fixed point $(H_c, \psi_c) = (\sqrt{3\alpha/\kappa}, \sqrt{6\alpha/\kappa})$.

It is to be stressed that although these considerations are based on the particular potential (5.17) for sake of illustration, the mechanism described accommodates any potential with the shape shown in Fig. 3. The value of the nonminimal coupling constant ξ is then adjusted, or reached during the running of ξ with the energy scale, as explained before, in such a way that the zero points of $V(\psi)$ become the corresponding critical values of the scalar field.

7. DISCUSSION AND CONCLUSIONS

The purpose of this paper is the exploration of unknown aspects of nonminimally coupled scalar field cosmologies; except for the (important) restriction to spatially flat FLRW spaces, this investigation is very general. Arbitrary scalar field potentials $V(\psi)$ and values of the coupling constant ξ of the scalar field to gravity are considered. The dynamics of the cosmological system are investigated in a self-consistent way, i.e., the equation of state of the cosmic fluid is not imposed *a priori*, but is rather derived together with the solution of the coupled Einstein–Klein–Gordon equations. Moreover, the source term in the right-hand side of the Einstein equations is taken to be the *full*, covariantly conserved, energy-momentum tensor of the scalar field including the geometric contributions originating by the nonminimal coupling of the scalar. This procedure avoids the questionable introduction of an effective time-dependent gravitational coupling $\kappa_{\text{eff}}(\tau)$ and the corresponding truncated, nonconserved energy-momentum tensor frequently encountered in the literature. This approach allows us to explore the richness and subtleties of the dynamics of nonminimally coupled scalar field cosmology. In different approaches, and for $\xi > 0$, this possibility is lost when $\kappa_{\text{eff}}(\tau)$ diverges, introducing an artificial barrier $\pm\psi_c$ to the values that ψ can assume and, correspondingly, a loss of generality since the solutions crossing this barrier are missed. Here it is shown that not only are the missing solutions restored, but they also play an important role and have an intriguing behavior; all this is due to the peculiarities of nonminimal coupling.

The exact critical solutions exhibit a universal character, as explained in Section 5; they are common to a wide class of potentials when the nonminimal coupling constant is promoted to the role of an adjustable parameter. One of these self-consistent solutions is of special physical interest, since it describes a nonsingular universe which spontaneously emerges from a contracting asymptotic de Sitter regime, contracts to a minimum nonzero size, and then spontaneously enters into an expanding de Sitter regime. Moreover, this behavior does not require the presence of a cosmological constant.

Several new properties emerge from the self-consistent dynamical approach: a dimensional reduction of the dynamical equations, using the Hubble function H and the scalar field ψ as variables, leads to the consequence that the dynamics of a spatially flat FLRW universe cannot exhibit chaotic behavior, thus settling a longstanding debate [24, 25]. Moreover, an involved topological structure of the phase space of the solutions appears: according to the form of the potential $V(\psi)$, dynamically forbidden regions may exist; they cannot be reached by the orbits of the solutions, and they are separated from the regions accessible to the orbits by a boundary $\mathcal{N}$. The accessible regions of the phase space consist of two two-dimensional sheets in the $(H, \psi,$

$\dot{\psi}$) space joining at the boundary $\mathcal{N}$ (for the values of V and ξ for which the latter exists). The phase space is topologically equivalent to $R_2 \times \{+, -\}$, with holes corresponding to the forbidden regions. It is argued that this peculiar topology may generate apparent chaotic regimes (reported for $k = 0$ FLRW spaces [24, 25]) in numerical integrations of the equations of scalar field cosmology.

Section 6 shows how the results of the previous sections, when extrapolated to the semiclassical framework, may lead to a revival of the issue [11, 38] of the birth of the universe from a primordial unstable Minkowski vacuum. Quantum vacuum fluctuations of the scalar field might carry the cosmological system through a classically forbidden region to its boundary $\mathcal{N}$ and to a classically allowed region. Specifically, the cosmological system might tunnel from the initial, classically allowed, empty, Minkowski space $(H, \psi) = (0, 0)$, which is an isolated unstable point of the boundary $\mathcal{N}$, to a classically allowed dynamical solution. The latter, which can be the critical solution discussed above, then acts as an initial data set to evolve the system toward an asymptotic expanding de Sitter regime. The merit of this semiclassical mechanism would be to lead continuously (except for a finite discontinuity in the pressure at the time of emergence of the classical solution) from the Minkowski space vacuum (at $\tau = 0$) to an asymptotic (at $\tau \to +\infty$) inflationary de Sitter regime. We emphasize that the nonminimal coupling of the scalar field is mandatory for the realization of this mechanism.

Finally, we mention that we have already extended most of the results to the case of non-spatially flat FLRW spaces. These universes do not allow for the dimensional reduction of the phase space of the solutions to two dimensions, and therefore chaotic behavior is not *a priori* prevented. In principle, chaos could appear due to even slight deviations from spatial flatness; a quantitative analysis of this topic is ongoing. We are also extending the previous considerations to the case of several interacting scalar fields; these results will be presented in a forthcoming publication.

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